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## LATTICES IN SOCIAL NETWORKS WITH INFLUENCE

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We present an application of lattice theory to the framework of influence in social networks. The contribution of the paper is not to derive new results, but to synthesize our existing results on lattices and influence. We consider a two-action model of influence in a social network in which agents have to make their yes-no decision on a certain issue. Every agent is preliminarily inclined to say either 'yes' or 'no', but due to influence by others, the agent's decision may be different from his original inclination. We discuss the relation between two central concepts of this model: influence function and follower function. The structure of the set of all influence functions that lead to a given follower function appears to be a distributive lattice. We also consider a dynamic model of influence based on aggregation functions and present a general analysis of convergence in the model. Possible terminal classes to which the process of influence may converge are terminal states (the consensus states and non trivial states), cyclic terminal classes and unions of Boolean lattices.

*Keywords:* Influence function; follower function; distributive lattice; aggregation function; convergence; terminal class.

*Subject Classification:* C7, D7.

### 1. Introduction

Influence phenomena attract research interest in several scientific areas, e.g., in economics, sociology and social psychology, where both experiments on influence and theoretical models are proposed, in political science, in management and organizational studies. One of the particularly rich and elegant approaches to influence is based on using social networks. Economists and sociologists are interested in the study of how the structure of networks affects opinion and consen-

sus formation, which the key players in a network are, or how important, central or influential an agent's neighbors are. Among numerous works along these lines one should mention measures of prestige (Katz [1953]), (eigenvector) centrality (Bonacich [1987]; Bonacich and Lloyd [2001]; Borgatti [2005]; Borgatti [2006]; Borgatti and Everett [2006]), and intercentrality by Ballester *et al.* [2006]. The seminal network interaction model of information transmission and opinion formation is presented in DeGroot [1974]. Several variations and generalizations of his model have been proposed for the last fifteen years, see, e.g., DeMarzo *et al.* [2003]; Golub and Jackson [2010]; Jackson [2008]; Krause [2000]; Lorenz [2005]. In the sociological literature, an interpersonal influence model is investigated in Friedkin [1991]; Friedkin and Johnsen [1990]; Friedkin and Johnsen [1997]. Surveys of different approaches to influence are given, e.g., in Grabisch and Rusinowska [2010c] and Rusinowska [2010]. For broad overviews of dynamic models of social influence we also refer to, e.g., Jackson [2008], Golub and Jackson [2010].

The present paper deals with a particular framework of influence introduced in Hoede and Bakker [1982] and studied, e.g., in Grabisch and Rusinowska [2009]; Grabisch and Rusinowska [2010a]; Grabisch and Rusinowska [2011a]; Grabisch and Rusinowska [2010b]. We consider a social network in which agents (players) may *influence* each others when making decisions. Each agent has an *inclination* to choose one of the actions, but due to influence by others, the *decision* of the agent may be different from his original inclination. Such a transformation from the agents' inclinations to their decisions is represented by an *influence function*. An important concept in the influence model is that of *follower* of a coalition, which is defined as a player who decides according to the inclination of the coalition in question whenever all members of the coalition are inclined to choose the same action. The original model of influence was a two-action model in which agents say either 'yes' or 'no'. In Grabisch and Rusinowska [2010a] we investigate some tools to analyze such framework. In particular, we study a (weighted) influence index of a coalition upon an individual and different influence functions. Also some generalizations of the two-action model are proposed. In Grabisch and Rusinowska [2010b] we generalize the yes-no model of influence to a multi-choice framework in which agents have an ordered set of possible actions. The influence model in which every player has a continuum of actions is investigated in Grabisch and Rusinowska [2011b]. In Grabisch and Rusinowska [2009] we study the relation between the yes-no influence model and command games proposed by Hu and Shapley [2003b]; Hu and Shapley [2003a] who use the command structure to model players' interaction relations by simple games. In particular, we show that our framework of influence is more general than the framework of the command games.

In two of our works on influence we show that some structures in the framework of influence are lattices. In Grabisch and Rusinowska [2011a] we consider, among other issues, the relation between two central concepts of the influence model: influence function and follower function. We deliver sufficient and necessary conditions for a function to be a follower function and show that the structure of the set of all

influence functions that lead to a given follower function is a distributive lattice. In Grabisch and Rusinowska [forthcoming] we analyze the decision process in which the mutual influence does not stop after one step but iterates. In particular, we propose a model based on aggregation functions, where the inclination of each agent is obtained as an aggregation of the inclinations of the others. We provide a general analysis of convergence in the aggregation model and find all terminal classes and states. We show that possible terminal classes to which the process of influence may converge are terminal states (the consensus states and non trivial states), cyclic terminal classes, and unions of Boolean lattices (called regular terminal classes).

The idea of applying lattice theory to social phenomena, in particular, to the issues of influence, is not really new and has been already considered in the previous literature. Monjardet [2003] lists numerous domains of mathematical social sciences where lattice theory plays a significant role, e.g., cluster analysis, data analysis, game theory, assessment theory, social choice and consensus theory. He provides numerous references of the related literature. Also Caspard *et al.* [2012] show that lattices are present in many fields of mathematical social sciences. The importance and usefulness of lattices for the analysis of influence is stressed, e.g., in Martin [2002]; Martin [2006], and in several other works that this author refers to. He considers algebraic structures for diffusion (of information, control or opinion) in social networks. As discussed by the author, it is possible to recreate an influence network (not observed directly) by examining a certain structure of a lattice of distribution of information. Even in case of incomplete data, if gaps are moderate, then there are some techniques to construct a lattice of information. Martin [2002] argues that some diffusion processes give structures that are closed under union but not intersection, and are therefore different from another prominent technique for algebraic analysis of network data, based on Galois lattices (structures that are closed under intersection but not union). Martin [2006] considers different algebraic techniques for the analysis of social data. Algebraic approaches to dichotomous data can be used as models of processes, as many social structures are algebraic and data produced by such structures have algebraic properties, or as data reduction and interpretation that do not correspond to particular social processes.

The *contribution of the present paper* is the synthesis of our existing results on lattices and influence (Grabisch and Rusinowska [2011a]; Grabisch and Rusinowska [forthcoming]), and not the derivation of new theorems. Following Martin and some others who confirm the importance of discussing models of influence and lattices, we would like to stress the interest of synthesizing our published results. While in the two related papers (Grabisch and Rusinowska [2011a]; Grabisch and Rusinowska [forthcoming]) we present and prove some results that link lattices with influence concepts, in none of these works we discuss the usefulness of the results to agents in choosing their influence techniques. The reasons why lattices present in our influence frameworks are useful for the influence analysis are different from the reasons discussed by other authors of the existing works on lattices and influence. It is partially due to the fact that we consider two specific

models of influence, and lattices help to answer specific ‘influence questions’ related to these models. We point out two particular issues. One of them concerns the choice of influence functions that lead to desirable follower functions, i.e., desirable set of followers. The second issue answers the question of how to aggregate the others’ opinions in order to avoid some bad effects of the dynamic influence process. We elaborate more on these issues in the concluding section.

The remaining part of the paper is the following. In Section 2.1 we recapitulate basic concepts of posets and lattices. Some basic notions of directed graphs and aggregation functions that we need in the last part of the paper are recapitulated in Section 2.2. The one-step model of influence is presented in Section 3. Section 4 concerns applications of lattice theory to the one-step model of influence. In Section 5 we present an application of lattices to the dynamic model of influence based on aggregation functions. In Section 6 we present a short conclusion and a discussion of the usefulness of lattice theory in our works on influence that are systematized in the present paper.

## 2. Basic Notions

### 2.1. Lattices and posets

We recall here the necessary material on finite posets and lattices for the sequel (see, e.g., Davey and Priestley [1990]).

Let  $N$  be a finite set. We write for two functions  $F, G : 2^N \rightarrow 2^N$  that  $F \leq G$  if  $F(S) \subseteq G(S)$  for all  $S \subseteq N$ . As usual,  $F = G$  means  $F(S) = G(S)$  for all  $S \subseteq N$ .

$F : 2^N \rightarrow 2^N$  is *isotone* or *monotone nondecreasing* if  $S \subseteq T \subseteq N$  implies  $F(S) \subseteq F(T)$ . If the first inclusion is reversed, then  $F$  is said to be *antitone* or *monotone nonincreasing*. A function is *monotone* if it is either isotone or antitone.

A *partially ordered set (poset)*  $(P, \leq)$  is a set  $P$  endowed with a partial order  $\leq$ , i.e., a binary relation being reflexive, antisymmetric and transitive. We recall that in any poset  $(P, \leq)$ ,  $Q \subseteq P$  is a *downset* of  $P$  if  $x \in Q$  and  $y \leq x$  imply  $y \in Q$ . The set of all downsets of  $P$  is denoted by  $\mathcal{O}(P)$ . For any two elements  $x, y \in L$ ,  $x$  is *covered by*  $y$  or  $y$  *covers*  $x$  (denoted by  $x \prec y$  or  $y \succ x$ ) if  $x < y$  and there is no  $z$  such that  $x < z < y$ .

A *lattice*  $L$  is a poset such that for any  $x, y \in L$  their least upper bound  $x \vee y$  and greatest lower bound  $x \wedge y$  always exist. In a finite lattice  $(L, \leq)$ , we denote by  $\top, \perp$  the *top* and the *bottom* elements of  $L$ . A lattice is said to be *distributive* if  $\vee, \wedge$  obey distributivity. A lattice is *autodual* if reversing the order relation, the same lattice is obtained (up to an isomorphism). A lattice is *Boolean* if it is isomorphic to some lattice of subsets  $(2^N, \subseteq)$ . Every Boolean lattice is distributive and autodual.

An element  $j \in L$  is *join-irreducible* if it is not the bottom element and it cannot be expressed as a supremum of other elements, or equivalently, if it covers only one element. The set of all join-irreducible elements is denoted by  $\mathcal{J}(L)$ . When the lattice is distributive, any element  $x \in L$  can be expressed in a unique way as an irredundant supremum of join-irreducible elements. This means that if  $L$

is distributive, it suffices to know only  $\mathcal{J}(L)$  to reconstruct all the lattice. More precisely,  $L$  is isomorphic to  $\mathcal{O}(\mathcal{J}(L))$ , the set of all downsets of  $\mathcal{J}(L)$  (Birkhoff's theorem).

The *height function*  $h$  on  $L$  gives the length of a longest chain from  $\perp$  to any element in  $L$ . The *height* of the lattice is  $h(\top)$ . A lattice is *ranked* if  $x \succ y$  implies  $h(x) = h(y) + 1$ . If a lattice is distributive, then it is ranked and the length of any maximal chain from  $\perp$  to  $\top$  is  $|\mathcal{J}(L)|$ .

A function  $\tau : (L, \leq) \rightarrow (L, \leq)$  is a *dual closure operator* if  $x \geq \tau(x)$  for all  $x \in L$ ,  $\tau$  is isotone, and  $\tau(\tau(x)) = \tau(x)$ .

## 2.2. Basic notions of digraphs and aggregation functions

For Section 5, we need some basic concepts on directed graphs and aggregation functions. For more details, see, e.g., Seneta [1973], Horn and Johnson [1985], Meyer [2000], Jackson [2008].

Let  $X = \{x_1, \dots, x_m\}$  be a finite set of states. We consider the  $m \times m$  transition matrix  $\mathbf{P} = [p_{ij}]_{i,j=1,\dots,m}$ , where  $p_{ij}$  is the probability that the next state  $s(t+1)$  is  $x_j$  knowing that the current state  $s(t)$  is  $x_i$ , i.e.,  $p_{ij} = \text{Prob}(s(t+1) = x_j \mid s(t) = x_i)$ . This matrix is row-stochastic.

We associate to  $\mathbf{P}$  its *transition directed graph (digraph)*  $\Gamma$ , whose set of nodes is the set of states  $X$ , and there is a directed edge from  $x_i$  to  $x_j$  if and only if  $p_{ij} > 0$ . In such a case we say that  $x_j$  is a successor of  $x_i$ .

A *path* in  $\Gamma$  is a sequence of nodes  $x_{j_1}, \dots, x_{j_k}$  such that  $x_{j_{l+1}}$  is a successor of  $x_{j_l}$  for  $l = 1, \dots, k-1$ . A *cycle* is a path for which the first and last nodes coincide, and its length is the number of edges.

A (*strongly*) *connected component* (called also a *class*) is any subset  $S$  of nodes such that there is a path from any node to any other node of  $S$ , and which is maximal for this property. Obviously a single node can be a class.

If the graph has only one class (which is then  $X$ ), then the matrix is said to be *irreducible*. A class is *transient* if there is an edge going outside it, otherwise the class is said to be *recurrent* or *terminal*. The set of all classes  $C_1, \dots, C_k$  forms a partition of  $X$ .

A terminal class  $C$  is said to be *periodic of period*  $p$  if the greatest common divisor of the length of all cycles in  $C$  is  $p$ . If  $p = 1$ , then we say that the class is *aperiodic*.

When the matrix is irreducible, if its (unique) class is aperiodic, we say that the matrix is aperiodic too. Usually, aperiodic (and therefore irreducible) matrices are called *ergodic*.

An  $n$ -place *aggregation function* is any mapping  $A : [0, 1]^n \rightarrow [0, 1]$  satisfying

- (1)  $A(0, \dots, 0) = 0$ ,  $A(1, \dots, 1) = 1$  (boundary conditions)
- (2) If  $\mathbf{x} \leq \mathbf{x}'$  then  $A(\mathbf{x}) \leq A(\mathbf{x}')$  (nondecreasingness).

Aggregation functions are well-studied and there exist many families of them,

e.g., all kinds of means (geometric, harmonic, quasi-arithmetic) and their weighted version, weighted ordered averages, any combination of minimum and maximum (lattice polynomials or Sugeno integrals), Choquet integrals, triangular norms, copulas; see Grabisch *et al.* [2009].

In the rest of the paper, by  $1_S = ((1_S)_1, \dots, (1_S)_n)$  with  $S \subseteq \{1, \dots, n\}$  we mean the characteristic vector of  $S$ , i.e.,  $(1_S)_i = 1$  if and only if  $i \in S$  and 0 otherwise.

### 3. The One-step Model of Influence

Let us recapitulate the model of influence originally introduced in Hoede and Bakker [1982], and next investigated e.g. in Grabisch and Rusinowska [2009]; Grabisch and Rusinowska [2010a]; Grabisch and Rusinowska [2011a].

We consider a social network with the set  $N := \{1, \dots, n\}$  of agents (players). The model is general and there is no restriction on the social network. Each agent  $k \in N$  is supposed to make a certain yes-no decision and has an inclination  $i_k$  either to say ‘yes’ (denoted by 1) or ‘no’ (denoted by 0). Let  $i = (i_1, \dots, i_n)$  denote an *inclination vector* and  $I := \{0, 1\}^n$  the set of all inclination vectors. Let for any  $S \subseteq N$ ,  $|S| \geq 1$ ,

$$I_S := \{i \in I \mid \forall k, j \in S, i_k = i_j\}, \quad (1)$$

i.e.,  $I_S$  is the set of all inclination vectors under which all members of coalition  $S$  have the same inclination. By  $i_S$  we denote the value  $i_k$  for some  $k \in S$ ,  $i \in I_S$ .

Agents may influence each other in the network, and due to these influences, the decision of an agent may be different from his (original) inclination. Formally, each inclination vector  $i \in I$  is transformed into a *decision vector*  $B(i) = ((B(i))_1, \dots, (B(i))_n)$ , where  $B : I \rightarrow I, i \mapsto B(i)$  is the *influence function*. The set of all influence functions is denoted by  $\mathcal{B}$ .

In this paper, we will use the set notation, i.e.,  $S$  will denote the set of yes-inclined agents and it corresponds to the inclination vector  $(1_S, 0_{N \setminus S})$ , i.e., to the  $n$ -vector  $i = (i_1, \dots, i_n)$  such that  $i_k = 1$  if  $k \in S$  and  $i_k = 0$  if  $k \in N \setminus S$ . Consequently,  $\bar{S} := N \setminus S$  is the set of players that are inclined to say ‘no’. If  $S$  corresponds to  $i$ , then  $B(i)$  will be denoted by  $B(S)$ , where  $B(S) \subseteq N$  is the set of agents whose decision is ‘yes’, and  $\overline{B(S)} := N \setminus B(S)$  is the set of agents whose decision is ‘no’. Consequently, an influence function  $B : I \rightarrow I$  can also be seen as a mapping from  $2^N$  to  $2^N$ .

Next, we define the concept of a *follower* which is one of the key concepts of the influence model. A follower of a given coalition is an agent who decides according to the inclination of the coalition in question in every situation in which all members of the coalition have the same inclination. Let  $B \in \mathcal{B}$ . The *follower function* under the influence function  $B$  is a mapping  $F_B : 2^N \rightarrow 2^N$  given by

$$F_B(S) := \{k \in N \mid \forall i \in I_S, (B(i))_k = i_S\}, \quad \forall S \subseteq N, S \neq \emptyset, \quad (2)$$

and  $F_B(\emptyset) := \emptyset$ .  $F_B(S)$  is the *set of followers of  $S$  under  $B$* . In other words, the follower function assigns to every coalition the set of followers of that coalition, i.e.,

all players that always decide according to the inclination of this (unanimous) coalition. The set of all follower functions is denoted by  $\mathcal{F}$ . In Grabisch and Rusinowska [2010a] we show that  $F_B$  is isotone and  $F_B(S) \cap F_B(T) = \emptyset$  whenever  $S \cap T = \emptyset$ .

One can check that, using the set notation, the follower function is defined as

$$F_B(S) = \bigcap_{S' \supseteq S} B(S') \cap \bigcap_{S' \subseteq N \setminus S} \overline{B(S')}, \quad \forall S \subseteq N, S \neq \emptyset, \quad (3)$$

and  $F_B(\emptyset) := \emptyset$ .

Assume  $F_B$  is not identically the empty set. The *kernel*  $\mathcal{K}(B)$  of influence function  $B$  is the set of ‘truly’ influential coalitions, i.e., the set of minimal coalitions having followers:

$$\mathcal{K}(B) := \{S \in 2^N \mid F_B(S) \neq \emptyset, \text{ and } S' \subset S \Rightarrow F_B(S') = \emptyset\}. \quad (4)$$

The kernel is well defined due to isotonicity of  $F_B$ .

#### 4. Applying Lattice Theory to the Influence Concepts

In Grabisch and Rusinowska [2011a] we study the exact relation between two concepts: influence function and follower function. While both an influence function  $B$  and a follower function  $F_B$  are mappings from  $2^N$  to  $2^N$ ,  $F_B$  should satisfy some conditions (like isotonicity) which need not be satisfied by  $B$ .

We define the mapping  $\Phi : \mathcal{B} \rightarrow (2^N)^{(2^N)}$  by  $B \mapsto \Phi(B) := F_B$ , i.e.,  $\Phi$  assigns to every influence function  $B$ , the follower function  $F_B$  under  $B$ . We have  $\Phi(\mathcal{B}) =: \mathcal{F}$ . This mapping is obviously neither a surjection nor an injection. Note that:

**Remark 1.** The function  $\Phi$  is not monotone (neither isotone nor antitone), so it fails to be a dual closure operator. Table 1 shows this fact. Clearly,  $B' \geq B$ , but  $\Phi(B)$  and  $\Phi(B')$  are incomparable.

Table 1. Remark 1 -  $\Phi$  is not monotone

$S$	$\emptyset$	1	2	3	12	13	23	123
$B(S)$	$\emptyset$	$\emptyset$	2	$\emptyset$	2	3	12	123
$\Phi(B)(S)$	$\emptyset$	$\emptyset$	2	$\emptyset$	2	3	12	123
$B'(S)$	$\emptyset$	$\emptyset$	12	2	12	3	12	123
$\Phi(B')(S)$	$\emptyset$	$\emptyset$	1	$\emptyset$	1	3	12	123

We deliver the sufficient and necessary conditions that a function from  $2^N$  to  $2^N$  is the follower function of a certain influence function. Moreover, if a function  $F : 2^N \rightarrow 2^N$  is a follower function, we find examples of  $B$ ’s such that  $F_B = F$ . We have the following result:

**Theorem 1 (Grabisch and Rusinowska [2011a]).** *A function  $F : 2^N \rightarrow 2^N$  is a follower function of some  $B \in \mathcal{B}$  (i.e.,  $F_B = F$ , or  $\Phi(B) = F$ ) if and only if it satisfies the following three conditions:*



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- (1)  $F(\emptyset) = \emptyset$ ;
- (2)  $F$  is isotone;
- (3) If  $S \cap T = \emptyset$ , then  $F(S) \cap F(T) = \emptyset$ .

Moreover, the smallest and greatest influence functions belonging to  $\Phi^{-1}(F)$  are respectively the influence functions  $\underline{B}_F$  and  $\overline{B}_F$ , defined by, in set notation:

$$\underline{B}_F(S) = F(S), \quad \overline{B}_F(S) = \overline{F(\overline{S})}, \quad \forall S \subseteq N. \quad (5)$$

We call these influence functions the lower and upper inverses of  $F$ .

We are interested in  $\Phi^{-1}(F)$  which, for a given follower function  $F$ , is the set of all possible influence functions that lead to that  $F$ . We determine the algebraic structure of  $\Phi^{-1}(F)$ . We already know that all elements of the inverse of  $F$  are between  $\underline{B}_F$  and  $\overline{B}_F$ , with the usual order  $\leq$  on functions. This means that  $(\Phi^{-1}(F), \leq)$  is a poset, which is a subset of  $([\underline{B}_F, \overline{B}_F], \leq)$ .

Let us write for simplicity:

$$D_S := \overline{B}_F(S) \setminus \underline{B}_F(S), \quad S \subseteq N. \quad (6)$$

An element of  $[\underline{B}_F, \overline{B}_F]$  can be denoted by the  $2^n$ -dim vector  $(T_\emptyset, \dots, T_N)$ , where  $T_S \subseteq D_S$  for each  $S \subseteq N$ . Using this notation,  $\underline{B}_F$  and  $\overline{B}_F$  are denoted by  $(\emptyset, \dots, \emptyset)$  and  $(D_\emptyset, \dots, D_N)$ , respectively. We have the following:

**Theorem 2 (Grabisch and Rusinowska [2011a]).** *Let  $B := (T_\emptyset, \dots, T_N) \neq \overline{B}_F$  be an element of  $\Phi^{-1}(F)$ . Then for any  $S \subseteq N$  such that  $D_S \setminus T_S \neq \emptyset$ , and any  $k \in D_S \setminus T_S$ ,  $B' := (T_\emptyset, \dots, T_S \cup \{k\}, \dots, T_N)$  is an element of  $\Phi^{-1}(F)$  if and only if one of the following conditions is NOT satisfied:*

- (1) For any  $S' \supset S$ ,  $k \in B(S')$ ;
- (2) For any  $S' \subseteq N \setminus S$ ,  $k \notin B(S')$ .

The main result, fully describing the structure of  $\Phi^{-1}(F)$ , is the following.

**Theorem 3 (Grabisch and Rusinowska [2011a]).** *For any  $F \in \mathcal{F}$ , the set  $\Phi^{-1}(F)$ , endowed with the usual ordering of functions, has the following properties:*

- (1) The greatest and least elements are  $\overline{B}_F$  and  $\underline{B}_F = F$ .
- (2) It is a lattice, with supremum and infimum given by, for any  $S \in 2^N$ :

$$\begin{aligned} (B \vee B')(S) &:= B(S) \cup B'(S) \\ (B \wedge B')(S) &:= B(S) \cap B'(S) \end{aligned}$$

- (3)  $\Phi^{-1}(F)$  is autodual, i.e.,  $(\Phi^{-1}(F), \leq)$  and  $(\Phi^{-1}(F), \geq)$  are isomorphic. The duality is expressed as follows: to each element  $B := (T_\emptyset, \dots, T_S, \dots, T_N)$  of  $\Phi^{-1}(F)$  corresponds the element  $B' := (D_N \setminus T_N, \dots, D_{\overline{S}} \setminus T_{\overline{S}}, \dots, D_\emptyset \setminus T_\emptyset)$ .
- (4) There are  $\sum_{S \subseteq N} |D_S|$  join-irreducible elements, one for each  $k \in D_S$ ,  $S \subseteq N$ , either of the form  $(k_S \emptyset)$  if this element belongs to  $\Phi^{-1}(F)$ , otherwise of the form  $(k_S k_{\overline{S}} \emptyset)$ , where the notation  $(k_S \emptyset)$  is a shorthand for  $(\emptyset, \dots, \emptyset, k, \emptyset, \dots, \emptyset)$ , where  $k$  is at position  $S$ , and similarly for  $(k_S k_{\overline{S}} \emptyset)$ .

(5) The lattice is distributive and its height is  $h = \sum_{S \subseteq N} |D_S|$ .

We would like to comment on the results (4) of Theorem 3. From lattice theory (see Section 2.1), we know that when the lattice is distributive, then it is enough to know its join-irreducible elements. Any element of the lattice can be written in a unique way as an (irredundant) supremum of join-irreducible elements. Hence, knowing only these elements permits to reconstruct the entire lattice. Moreover, we can generate all elements of the lattice in the following way. Let  $\mathcal{J}$  denote the set of its join-irreducible elements, with the same order relation  $\leq$ . We consider an antichain  $\mathcal{A}$  of  $\mathcal{J}$ , i.e., a subset of  $\mathcal{J}$  such that any two elements  $B, B'$  of  $\mathcal{A}$  are incomparable, which means that neither  $B \leq B'$  nor  $B' \leq B$  hold. Then  $B := \bigvee_{B' \in \mathcal{A}} B'$  is an element of the lattice  $\Phi^{-1}(F)$ , and generating the entire lattice amounts to generating all antichains of  $\mathcal{J}$ . We can check whether  $(k_S \emptyset)$  is an element of  $\Phi^{-1}(F)$  by using Theorem 2 with  $(T_\emptyset, \dots, T_N) = (\emptyset, \dots, \emptyset)$ , i.e., we only need to check if one of the two following conditions fails:

- (1)  $\forall S' \supset S, k \in F(S')$ ;
- (2)  $\forall S' \subseteq N \setminus S, k \notin F(S')$ .

Note that if  $(k_S \emptyset)$  is not an element of  $\Phi^{-1}(F)$ , then necessarily  $(k_{\bar{S}} \emptyset)$  is.

These results are illustrated in the following example, borrowed from Grabisch and Rusinowska [2011a].

**Example 1.** Let  $n = 3$  and the function  $F$  be defined as in Table 2. The upper and lower inverses are given in Table 3. We compute the join-irreducible elements of  $\Phi^{-1}(F)$ . The sets  $D_S$  are given in Table 4.

Table 2. Example 1 - Function  $F$

$S$	$\emptyset$	1	2	3	12	13	23	123
$F(S)$	$\emptyset$	$\emptyset$	2	$\emptyset$	2	3	12	123

Table 3. Example 1 - The upper and lower inverses

$i$	$\emptyset$	1	2	3	12	13	23	123
$\overline{B}_F i$	$\emptyset$	3	12	13	123	13	123	123
$\underline{B}_F i$	$\emptyset$	$\emptyset$	2	$\emptyset$	2	3	12	123

The set  $\mathcal{J}$  of join-irreducible elements contains 8 elements, given by:

- For  $S = 1, k = 3$ :  $(3_1 \emptyset)$  belongs to  $\Phi^{-1}(F)$ , so it is a join-irreducible element.
- For  $S = 2, k = 1$ :  $(1_2 \emptyset)$  belongs to  $\Phi^{-1}(F)$ , so it is a join-irreducible element.

Table 4. Example 1 - The sets  $D_S$ 

$S$	$\emptyset$	1	2	3	12	13	23	123
$D_S$	$\emptyset$	3	1	13	13	1	3	$\emptyset$

- For  $S = 3, k = 1$ :  $(1_3\emptyset)$  belongs to  $\Phi^{-1}(F)$ , so it is a join-irreducible element.
- For  $S = 3, k = 3$ :  $(3_3\emptyset)$  belongs to  $\Phi^{-1}(F)$ , so it is a join-irreducible element.
- For  $S = 12, k = 1$ :  $(1_{12}\emptyset)$  does not belong to  $\Phi^{-1}(F)$ , so  $(1_{12}1_3\emptyset)$  is a join-irreducible element.
- For  $S = 12, k = 3$ :  $(3_{12}\emptyset)$  does not belong to  $\Phi^{-1}(F)$ , so  $(3_{12}3_3\emptyset)$  is a join-irreducible element.
- For  $S = 13, k = 1$ :  $(1_{13}\emptyset)$  does not belong to  $\Phi^{-1}(F)$ , so  $(1_{13}1_2\emptyset)$  is a join-irreducible element.
- For  $S = 23, k = 3$ :  $(3_{23}\emptyset)$  does not belong to  $\Phi^{-1}(F)$ , so  $(3_{23}3_1\emptyset)$  is a join-irreducible element.

From this we can generate all elements of  $\Phi^{-1}(F)$ . For example,  $\{(1_{12}1_3\emptyset), (3_1\emptyset), (1_2\emptyset)\}$  is an antichain of  $\mathcal{J}$ , and therefore  $(3_11_21_31_{12}) = (\emptyset, 3, 1, 1, 1, \emptyset, \emptyset, \emptyset)$  is an element of  $\Phi^{-1}(F)$ .

Let us finish this section with a short remark concerning atomicity of the lattice  $\Phi^{-1}(B)$ . As shown by Example 1, not all join-irreducible elements are atoms. More precisely, all elements of the form  $(k_S\emptyset)$  are atoms, but elements of the form  $(k_S k_{\bar{S}}\emptyset)$  are not atoms, and these elements appear as soon as  $(k_S\emptyset)$  does not belong to  $\Phi^{-1}(B)$ . It seems difficult then to investigate in the general case when this happens.

## 5. Lattices in the Iterated Model of Influence

Next, we generalize the yes-no model of influence in a social network with a single step of mutual influence to a framework with iterated influence.

Given the current set  $S$  of ‘yes’-voters, there is a certain probability  $b_{S,T}$  that the set of ‘yes’-voters after one step of influence is  $T$ . If we assume that the process of influence may iterate (several rounds in the discussion), we obtain a *stochastic process* which depicts the evolution of the coalition of ‘yes’-agents along time. We assume that:

- (1) The process is Markovian, i.e., the probability  $b_{S,T}$  depends on  $S$  (the present situation) and  $T$  (the future situation), and not on the whole history.
- (2) The process is stationary, i.e.,  $b_{S,T}$  is constant over time.

States of this finite Markovian process are therefore all subsets  $S \subseteq N$ , representing the set of ‘yes’-agents; its *transition matrix*  $\mathbf{B} := [b_{S,T}]_{S,T \subseteq N}$  is a  $2^n \times 2^n$  row-stochastic matrix.

In Grabisch and Rusinowska [forthcoming] we propose a model of influence based on aggregation functions. In other words, to each agent  $k \in N$  we associate an aggregation function  $A_k$  which specifies the way agent  $k$  modifies his opinion from the opinion of the other agents and himself. More precisely, the output  $A_k(1_S) \in [0, 1]$  of agent  $k$ 's aggregation function is the probability that  $k$  says 'yes' after one step of influence, when the current situation is  $1_S$  (i.e., the agents in  $S$  say 'yes' and the others say 'no'). We denote by  $\mathbf{A} := (A_1, \dots, A_n)$  the vector of aggregation functions. Specifically, supposing that  $S$  is the set of agents saying 'yes', we compute  $\mathbf{A}(1_S) = (A_1(1_S), \dots, A_n(1_S))$ , that is, vector  $\mathbf{A}(1_S)$  indicates the probability of each agent to say 'yes' after influence.

By using the model based on aggregation functions, we find the Markovian influence model. In particular, if we assume that the probabilities to say 'yes' are independent among agents, then the probability of transition from the yes-coalition  $S$  to the yes-coalition  $T$  is

$$b_{S,T} = \prod_{k \in T} A_k(1_S) \prod_{k \notin T} (1 - A_k(1_S)) \quad (7)$$

which determines  $\mathbf{B}$ . To calculate the probability  $b_{S,T}$ , we then need to multiply the probabilities of every agent in  $T$  to say 'yes' and of every agent outside  $T$  to say 'no' after one step of influence, assuming that all agents in  $S$  and only them say 'yes' in the current situation.

It follows that deterministic models correspond to aggregation functions satisfying  $A_k(1_S) \in \{0, 1\}$  for all  $k \in N$ , hence they reduce to Boolean functions  $B_k : 2^N \rightarrow \{0, 1\}$  which are nondecreasing and nonconstant. Conversely, any deterministic influence model  $B : 2^N \rightarrow 2^N$  satisfying  $B(\emptyset) = \emptyset$ ,  $B(N) = N$  and being nondecreasing is a particular aggregation model, with nondecreasing nonconstant Boolean functions  $B_1, \dots, B_n$  defined by  $B_k(1_S) = 1$  if  $B(S) \ni k$ , and 0 otherwise.

We provide a general analysis of convergence of influence functions based on aggregation functions. We have the following result:

**Theorem 4 (Grabisch and Rusinowska [forthcoming]).** *Suppose  $\mathbf{B}$  is obtained from an aggregation model, with aggregation functions  $A_1, \dots, A_n$ . Then terminal classes are:*

- (1) *either singletons  $\{S\}$ ,  $S \in 2^N$ ,*
- (2) *or cycles of nonempty sets  $\{S_1, \dots, S_k\}$  of any length  $2 \leq k \leq \binom{n}{\lfloor n/2 \rfloor}$  (and therefore they are periodic of period  $k$ ) with the condition that all sets are pairwise incomparable (by inclusion),*
- (3) *or collections  $\mathcal{C}$  of nonempty sets with the property that  $\mathcal{C} = \mathcal{C}_1 \cup \dots \cup \mathcal{C}_p$ , where each subcollection  $\mathcal{C}_j$  is a Boolean lattice  $[S_j, S_j \cup K_j]$ ,  $S_j \neq \emptyset$ ,  $S_j \cup K_j \neq N$ , and at least one  $K_j$  is nonempty.*

*We call terminal states those terminal classes of the first type, cyclic terminal classes those of the second type, and regular terminal classes those terminal classes of the*

third type. Regular terminal classes formed by a single Boolean lattice  $[S, S \cup K]$  are called Boolean terminal classes.

The terminal states can represent the two consensus situations ( $\{N\}$  where everybody in the society says ‘yes’, and  $\{\emptyset\}$  where everybody says ‘no’), and situations of the polarized society (where the individuals within the class say ‘yes’ while the others say ‘no’). These terminal states which are not the consensus states are called nontrivial terminal states. In the second type of terminal classes, states form a cycle of nonempty sets, and the period of the class determines after how many steps a state is reached again. A Boolean terminal class  $[S, S \cup K]$  can be interpreted as a situation where agents in  $S$  finally decided to say ‘yes’ and agents outside  $S \cup K$  finally decided to say ‘no’, while the agents in  $K$  change their opinion non-deterministically forever. For a general case of regular terminal classes, i.e., with more than one interval, the interpretation is more complex and depends on the transitions between the intervals.

Examples 2, 3, and 4 show the existence of (nontrivial) terminal states, cyclic classes, and regular terminal classes, respectively.

**Example 2.** Consider  $N = \{1, 2, 3\}$  and the following aggregation functions:

$$\begin{aligned} \mathbf{A}(1\ 0\ 0) &= (1\ x\ 0) & \mathbf{A}(0\ 1\ 0) &= (x\ 1\ 0) & \mathbf{A}(0\ 0\ 1) &= (0\ 0\ 1) \\ \mathbf{A}(1\ 1\ 0) &= (1\ 1\ 0) & \mathbf{A}(1\ 0\ 1) &= (x\ 1\ 1) & \mathbf{A}(0\ 1\ 1) &= (1\ x\ 1) \end{aligned}$$

with arbitrary  $0 < x < 1$ . The digraph for the Markov chain is presented in Fig. 1. Apart from the consensus states  $\{123\}$  and  $\{\emptyset\}$ , we have also nontrivial terminal states  $\{12\}$  and  $\{3\}$ .

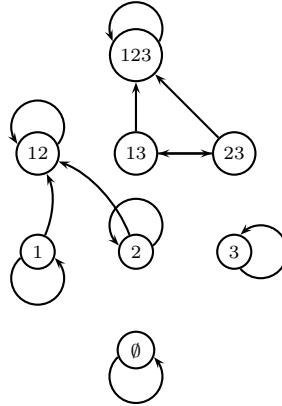


Fig. 1. Example of (nontrivial) terminal states

**Example 3.** Consider  $N = \{1, 2, 3\}$  and the following aggregation functions:

$$\begin{aligned} \mathbf{A}(1\ 0\ 0) &= (0\ 1\ 0) & \mathbf{A}(0\ 1\ 0) &= (0\ 0\ 1) & \mathbf{A}(0\ 0\ 1) &= (1\ 0\ 0) \\ \mathbf{A}(1\ 1\ 0) &= (0\ 1\ 1) & \mathbf{A}(1\ 0\ 1) &= (1\ 1\ 0) & \mathbf{A}(0\ 1\ 1) &= (1\ 0\ 1). \end{aligned}$$

The digraph for the Markov chain is given in Fig. 2. Then  $\{1, 2, 3\}$  and  $\{12, 23, 13\}$  are cyclic classes.

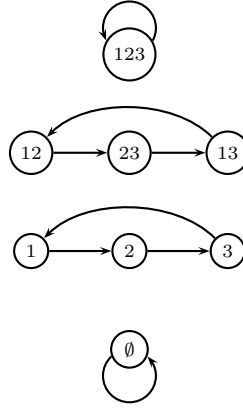


Fig. 2. Example of cyclic classes

**Example 4.** Consider  $N = \{1, 2, 3\}$  and the following aggregation functions:

$$\begin{aligned} A_1(1\ 0\ 0) &= 1 & A_2(1\ 0\ 0) &= 0.5 & A_3(1\ 0\ 0) &= 0 \\ A_1(0\ 1\ 0) &= 0 & A_2(0\ 1\ 0) &= 0.5 & A_3(0\ 1\ 0) &= 0 \\ A_1(0\ 0\ 1) &= 0 & A_2(0\ 0\ 1) &= 0.5 & A_3(0\ 0\ 1) &= 0.5 \\ A_1(1\ 1\ 0) &= 1 & A_2(1\ 1\ 0) &= 0.5 & A_3(1\ 1\ 0) &= 0 \\ A_1(1\ 0\ 1) &= 1 & A_2(1\ 0\ 1) &= 0.5 & A_3(1\ 0\ 1) &= 0.5 \\ A_1(0\ 1\ 1) &= 1 & A_2(0\ 1\ 1) &= 0.5 & A_3(0\ 1\ 1) &= 1. \end{aligned}$$

This gives the digraph for the Markov chain as presented in Fig. 3. Clearly,  $\{1, 12\}$  is a regular terminal class.

Regular terminal classes can be periodic, as Example 5 shows.

**Example 5.** Consider  $N = \{1, 2, 3\}$  and the following aggregation functions:

$$\begin{aligned} \mathbf{A}(1\ 0\ 0) &= \mathbf{A}(1\ 1\ 0) = (0\ x\ 1) \\ \mathbf{A}(0\ 0\ 1) &= \mathbf{A}(0\ 1\ 1) = (1\ x\ 0) \\ \mathbf{A}(0\ 1\ 0) &= \mathbf{A}(0\ 0\ 0) = (0\ 0\ 0) \\ \mathbf{A}(1\ 0\ 1) &= \mathbf{A}(1\ 1\ 1) = (1\ 1\ 1) \end{aligned}$$

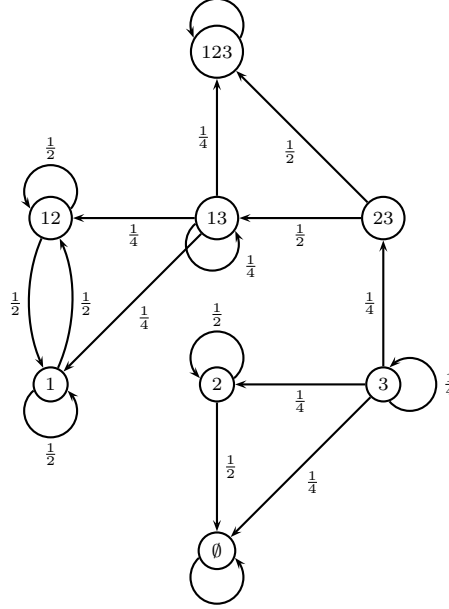


Fig. 3. Example of a regular terminal class

with arbitrary  $0 < x < 1$ . Then  $\{1, 3, 12, 23\}$  forms a periodic terminal class of period 2 (see Fig. 4, left). Now consider the following aggregation functions:

$$\mathbf{A}(100) = \mathbf{A}(110) = (001)$$

$$\mathbf{A}(001) = (1x0)$$

$$\mathbf{A}(010) = \mathbf{A}(000) = (000)$$

$$\mathbf{A}(101) = \mathbf{A}(011) = \mathbf{A}(111) = (111)$$

with arbitrary  $0 < x < 1$ . Then  $\{1, 3, 12\}$  is a periodic class of period 2 with 3 sets (see Fig. 4, right).

## 6. Concluding Remarks

In this paper, some selected results on the model of influence that are related to lattices have been presented. First of all, we should clarify that the relation between the two works (Grabisch and Rusinowska [2011a]; Grabisch and Rusinowska [forthcoming]) lies in the framework of influence that we consider and in the importance of lattices for the influence analysis, rather than in the type of lattice structures that we find. The first work deals with the one-step model of influence, and the second one concerns the dynamic and extended version of the previous work. It appears that the structure of the set of all influence functions that lead to a given follower function is a distributive lattice. Moreover, in the iterated model of influence, for influence functions based on aggregation, some terminal classes are unions

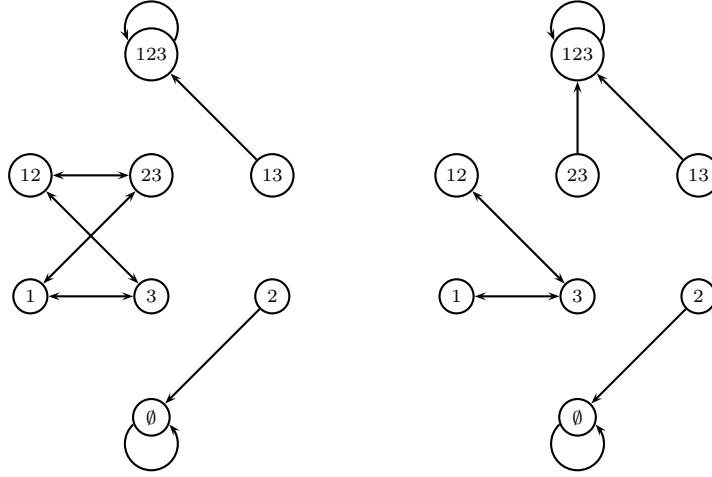


Fig. 4. Examples of a periodic terminal class

of Boolean lattices. Hence, in both models, lattices are “present” and appear to be useful as discussed below.

As introduced in Section 3, the follower function assigns to every coalition the set of followers of that coalition. Some individuals might be interested in having some other concrete agents as their followers or as followers of specific coalitions that these individuals belong to. In other words, such individuals might be interested in specific follower functions, and therefore they should learn which ‘influence techniques’ (i.e., which influence functions) lead to the specific follower functions. It would be therefore helpful to them to know the set of all influence functions that lead to the particular follower function. This set is  $\Phi^{-1}(F)$  and since it appears to be a distributive lattice, it is enough to know its join-irreducible elements. The essential help for this problem comes from the recapitulated Theorem 3 proved in Grabisch and Rusinowska [2011a].

In numerous dynamic models of influence, possible terminal classes in which the process of influence might end up are reduced only to the consensus states. In Grabisch and Rusinowska [forthcoming] we show that if the individuals are allowed to aggregate opinions arbitrarily, not only by using the weighted arithmetic means, then other terminal classes may appear. As a result, the society may be polarized into ‘yes’- and ‘no’-individuals, or there may be a cycle of nonempty sets, or the society may be divided into three types of individuals: ‘yes’-individuals, ‘no’-individuals, and the ones that change their opinion non-deterministically forever. This is known due to the recapitulated Theorem 4 proved in Grabisch and Rusinowska [forthcoming]. One can imagine that some individuals, or even the society as a whole, would like to avoid some of these terminal classes. For instance, the lattice structure is not a particularly desirable terminal class, as it represents situations in which there exist some individuals that permanently change their opinions. Under such ‘lattice



structure situations', neither reaching the consensus nor having two stable groups of individuals with the opposite opinions is possible. Consequently, the decision process, for instance, in committees and all kinds of decision-making bodies, seems to be more difficult under the lattice structure. Our results show that it is possible to 'manipulate the future' by not using some ways to aggregate opinions. For (sufficient and necessary) conditions to avoid the particular types of terminal classes in the dynamic model of influence, we refer to Grabisch and Rusinowska [forthcoming].

Obviously, there are several possibilities to continue this line of research, both by considering other concepts in the framework of influence as well as by analyzing social phenomena different from the social influence.

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